View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Elsevier - Publisher Connector

Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Algebra 268 (2003) 419–443
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Endomorphic presentations of branch groups

Laurent Bartholdi

Department of Mathematics, University of California, Berkeley, CA 94720, USA

Received 24 July 2000

Communicated by Alexander Lubotzky

Abstract

We introduce "endomorphic presentations," or L-presentations: group presentations whose relations are iterated under a set of substitutions on the generating set, and show that a broad class of groups acting on rooted trees admit explicitly constructible finite L-presentations, generalising results by Igor Lysionok and Said Sidki.
© 2003 Elsevier Inc. All rights reserved.

Keywords: Fractal group; Branch group; L-system; Group presentation; Schur multiplier

1. Introduction

In the early 80s, Rostislav Grigorchuk defined a group, G, endowed with many interesting properties: it is a finitely generated, infinite, torsion group; it has intermediate growth; it has a solvable word problem; it has finite width; etc. There are connections of G to innumerably many branches of mathematics: random walks on graphs, Hecke operators, classification of finite-rank Lie algebras, cryptography, etc.

Already in his early papers [Gri84], Rostislav Grigorchuk showed that G is not finitely presentable. However, Igor Lysionok obtained in [Lys85] a recursively defined, infinite set of relators for G, obtained by iterating a simple letter substitution on a finite set of relators (see Theorem 4.5):

The Grigorchuk group G admits the following presentation:

$$G = \left\langle a, c, d \,\middle|\, \sigma^i(a)^2, \sigma^i(ad)^4, \sigma^i(adacac)^4 \ (i \geqslant 0) \right\rangle,$$

where $\sigma : \{a, c, d\}^* \to \{a, c, d\}^*$ is defined by $\sigma(a) = aca$, $\sigma(c) = cd$, $\sigma(d) = c$.

E-mail address: laurent@math.berkeley.edu.

0021-8693/$ – see front matter © 2003 Elsevier Inc. All rights reserved.
doi:10.1016/S0021-8693(03)00268-0

Rostislav Grigorchuk then used this result to construct a finitely presented amenable but not elementary-amenable group [Gri98], thus answering negatively to an old question by Mahlon Day [Day57]: "Can every amenable finitely presented group be obtained from finite and abelian groups using exact sequences and unions?" He also proved the independence of Lysionok's relators, by explicitly computing the Schur multiplier $H_2(G,\mathbb{Z})$ in [Gri99].

A (friendly) competitor of the Grigorchuk group is the Gupta–Sidki group $\overline{\overline{\Gamma}}$, which is also a finitely generated infinite torsion group of subexponential growth. Both groups share other properties, as well—see, for instance, [BG00,BG02] where they are studied simultaneously. Said Sidki described in [Sid87] a general method yielding recursive presentations of such groups, and for $\overline{\overline{\Gamma}}$ derived an explicit, if somewhat lengthy, presentation.

Narain Gupta also followed a completely different path in obtaining recursively presented torsion groups—namely, he started by defining a presentation, that is recursive but presents no explicit regularity like the presentations considered here, and then proves that the associated group is infinite, torsion, and finitely generated [Gup84].

In this paper, we define a general class of presentations, which we call *endomorphic* or *L-presentations*. As a first approximation, they are given by a generating set, some initial relations, and word substitution rules that produce more relations.

We start by deriving some of their properties, and give explicit presentations for $\overline{\overline{\Gamma}}$ and other contracting branch groups (see Definitions 3.3 and 3.4; the main property of a *branch group* is that it contains a subgroup K containing a copy of $K_1 \cong K^d$ for some $d \geqslant 2$, all inclusions having finite index. It is *contracting* if there is a metric on K contracted, up to an additive constant, by the projections $K_1 \to 1^i \times K \times 1^{d-i-1}$). Our main result on groups acting on rooted trees is the following (see Theorems 3.1 and 3.2):

Theorem 1.1. *Let G be a finitely generated, contracting, semi-fractal, regular branch group. Then G is finitely L-presented. However, G is not finitely presented.*

The Schur multiplier of G has the form $A \oplus B^\infty$ for finite-rank groups A, B.

Definition 1.2. An *L-presentation* is an expression of the form

$$L = \langle S \mid Q \mid \Phi \mid R \rangle,$$

where S is an alphabet (i.e., a set of symbols), $Q, R \subset F_S$ are sets of reduced words (where F_S is the free group on S), and Φ is a set of free group homomorphisms $\phi : F_S \to F_S$.

L is *finite* if S, Q, Φ, R are finite. It is *ascending* if Q is empty. It is *injective* if, furthermore, all $\phi \in \Phi$ are injective.

L gives rise to a group G_L defined as

$$G_L = F_S \Big/ \Big\langle Q \cup \bigcup_{\phi \in \Phi^*} \phi(R) \Big\rangle^{\#},$$

where $\langle \cdot \rangle^{\#}$ denotes normal closure and Φ^* is the monoid generated by Φ, i.e., the closure of $\{1\} \cup \Phi$ under composition.

As is customary, we shall identify the presentation L and the group G_L it defines, and write G for both.

The name "L-presentation" comes both as a homage to Igor Lysionok who discovered such a presentation for the Grigorchuk group G (see Theorem 4.5) and as a reference to "L-systems" as defined by Aristid Lindenmayer [Lin73] in the early 70s (see [RS80]), used to model biological growth phenomena.

L-presentations are defined here for the first time; however, they appear implicitly in all situations where an infinite-index subgroup with a nice-enough transversal (say, isomorphic to $\mathbb{Z}$) is considered. A typical example is Wilhelm Magnus' Freiheitssatz [CM82, II.5], where a 1-relator group is shown to be the HNN extension of an infinitely presented finite L-presentation, which in turn is shown to be again a 1-relator group.

1.1. Symmetric groups

The purpose of L-presentations is to encode in homomorphisms $\phi \in \Phi$ some regularity of the presentation. Consider, for instance, the presentations of finite symmetric groups. It is well known that the following is a presentation of $\mathfrak{S}_n$, the symmetric group on n objects (see [Bur97,Moo97] for its first occurrences in literature and [Ser93] for other presentations):

$$\begin{aligned}\mathfrak{S}_n = \big\langle \sigma_1, \dots, \sigma_{n-1} \,\big|\, & \sigma_i^2 \text{ whenever } 1 \leqslant i \leqslant n-1, \\ & (\sigma_i\sigma_{i+1})^3 \text{ whenever } 1 \leqslant i \leqslant n-2, \\ & (\sigma_i\sigma_j)^2 \text{ whenever } 1 \leqslant i < j-1 \leqslant n-2\big\rangle.\end{aligned}$$

A shorter ascending L-presentation with the same generators can be obtained if one lets the symmetric group act on itself by conjugation; there remain only 3 orbits of relators under this action. To the point, consider the set $P = \{(1, \dots, n), (1, 2), (3, \dots, n)\}$ generating $\mathfrak{S}_n$. For each $p \in P$, let it act as ϕ_p on the free group $F_{\sigma_1,\dots,\sigma_{n-1}}$ in such a way that this action is a lift of the action of $\mathfrak{S}_n$ by conjugation on itself, and such that if $\sigma_i^p = \sigma_j$ then $\phi_p(\sigma_i) = \sigma_j$—a simple way of selecting such a ϕ_p is to pick for each σ_i a word W over $\{\sigma_1, \dots, \sigma_{n-1}\}$ of minimal length representing σ_i^p, and setting $\phi_p(\sigma_i) = W$, extended by concatenation. We then obtain

$$\mathfrak{S}_n = \big\langle \sigma_1, \dots, \sigma_{n-1} \,\big|\, \big|\, \{\phi_p\}_{p\in P} \,\big|\, \sigma_1^2, (\sigma_1\sigma_2)^3, (\sigma_1\sigma_3)^2 \big\rangle.$$

Indeed, all relations σ_i^2 and $(\sigma_i\sigma_{i+1})^3$ are obtained as $\phi_{(1,\dots,n)}^{i-1}(\sigma_1^2)$ and $\phi_{(1,\dots,n)}^{i-1}((\sigma_1\sigma_2)^3)$, and all relations $(\sigma_i\sigma_j)^2$ are obtained as $\phi_{(1,\dots,n)}^{i-1}\phi_{(3,\dots,n)}^{j-i-2}((\sigma_1\sigma_3)^2)$. Conversely, all $\phi(r)$ are relations for $\phi \in \{\phi_p\}^*$ and r a relation, since the ϕ are endomorphisms.

Using the same reasoning, one can obtain an ascending L-presentation of $\mathfrak{S}_n$ with only two relators, if one allows more generators:

$$\mathfrak{S}_n = \big\langle \sigma_{1,2}, \sigma_{1,3}, \dots, \sigma_{n-1,n} \,\big|\, \big|\, \phi_{(1,2)}, \phi_{(1,\dots,n)} \,\big|\, \sigma_{1,2}^2, \sigma_{1,2}\sigma_{2,3}\sigma_{1,3}\sigma_{2,3} \big\rangle,$$

where $\sigma_{i,j}$ should be interpreted as the transposition (i, j), and $\phi_p(\sigma_{i,j}) = \sigma_{i^p, j^p}$.

(This regularity in the presentation is reflected by the fact that $H_2(\mathfrak{S}_n, \mathbb{Z}) = \mathbb{Z}/2$ is very small—see Section 2.3.)

Problem 1.3. Does there exist a bound A such that all symmetric groups can be defined by an ascending L-presentation $\langle S| \mid \Phi| R\rangle$ of total length $|S| + |\Phi| + |R| < A$?

1.2. *Other examples*

Another example is given by presentations of the free Abelian group $\mathbb{Z}^n$:

$$\mathbb{Z}^n = \bigl\langle x_1, \dots, x_n \bigm| [x_i, x_j]\ \forall i, j \in \{1, \dots, n\}\bigr\rangle.$$

It can be expressed with fewer relators as

$$\mathbb{Z}^n = \bigl\langle x_1, \dots, x_n \bigm| \bigm| \phi_1, \phi_2 \bigm| [x_1, x_2]\bigr\rangle,$$

with $\phi_1 : x_1 \mapsto x_2, x_2 \mapsto x_3, \dots, x_n \mapsto x_1$ and $\phi_2 : x_1 \mapsto x_1, x_2 \mapsto x_3, x_3 \mapsto x_4, \dots, x_n \mapsto x_2$. Of course, the main interest of L-presentations is to encode groups that do not even have a finite presentation: consider for instance the group $\mathfrak{S}_\infty\mathbb{Z}$ of permutations of $\mathbb{Z}$ that act like a translation outside a finite interval. It is generated by $\sigma = (1, 2)$ and $\tau : n \mapsto n + 1$:

$$\begin{aligned}
\mathfrak{S}_\infty\mathbb{Z} &= \bigl\langle \tau, \sigma \bigm| \sigma^2, \bigl[\sigma, \tau^n\bigr]^2\ \forall n \geqslant 2, [\sigma, \tau]^3\bigr\rangle \\
&= \bigl\langle \tau, \sigma, \overline{\sigma} \bigm| \sigma\overline{\sigma} \bigm| \phi \bigm| \sigma^2, [\sigma, \tau]^3, [\sigma, \overline{\sigma}]^{\tau^2}\bigr\rangle \\
&= \bigl\langle \tau, \sigma, \overline{\tau} \bigm| \tau^{-1}\overline{\tau} \bigm| \psi \bigm| \sigma^2, [\sigma, \tau]^3, \bigl[\sigma, \sigma^{\tau\overline{\tau}}\bigr]\bigr\rangle
\end{aligned}$$

with $\phi(\overline{\sigma}) = \overline{\sigma}^\tau$ and $\psi(\overline{\tau}) = \tau\overline{\tau}$, both preserving the other generators σ and τ. The extra generators $\overline{\sigma}$ and $\overline{\tau}$ are just convenient copies of the generators.

1.3. *Outline*

The paper is organised as follows. Section 2 contains group-theoretical results on L-presentations. Section 3 describes the main result of this paper, namely that all finitely generated regular branch groups have a finite L-presentation. Section 4 describes the L-presentations of the 5 “testbed” branch groups introduced in [BG00].

1.4. *Notations*

For me, g^h denotes $h^{-1}gh$, and the expression $g^{\sum n_i h_i}$ means $\prod h_i^{-1} g^{n_i} h_i$. The commutator $[g, h]$ is $g^{-1}h^{-1}gh$, and X^* is the monoid generated by X. The normal closure of X in G is written $\langle X\rangle^\#$, the normal subgroup of G *normally generated* by X.

2. Group-theoretical properties

In this section, we are interested in the following questions:

- which group-theoretical constructions preserve the property of having a finite L-presentation?
- which groups admit a finite L-presentation?

We shall say a group is *finitely L-presented* if it admits a finite L-presentation.

Remark 2.1. There are finitely L-presented groups that, for some imposed generating set, do not admit a finite L-presentation. This is in contrast with finitely presented groups, for which admitting a finite presentation is independent of the choice of generators—that property is even invariant under quasi-isometries.

For instance, consider the "lamplighter group" of Theorem 4.1, with its finite L-presentation. This group G does not have a finite L-presentation with generators $\{a, t\}$, as can be seen by a careful study of endomorphisms of $\mathbb{F}_2$.

Proposition 2.2. *Let G admit a finite ascending L-presentation, and let S' be a finite generating set of G. Then G admits a finite ascending L-presentation with generators S'.*

Proof. The standard proof that being finitely presented involves Tietze transformations, and extends to L-presentations. One changes S into S' by a finite number of "Tietze moves," which either replace a generator by a product or quotient of generators, or add or delete a generator s along with the relator s.

For an L-presentation $\langle S \| \Phi | R\rangle$, the operations are as follows: if the move was to replace the generator s by $s' = st$, one replaces all instances of s by $s't^{-1}$ in R and the images of $\phi \in \Phi$, modifying them by $\phi(s') = \phi(st)$.

If the move was addition of a generator s to S and R, one extends all $\phi \in \Phi$ by $\phi(s) = 1$. If the move was deletion of s from S and R, one deletes all instances of s in the images of all $\phi \in \Phi$, and adds $\phi(s)$ to R. $\square$

2.1. Embeddings

We start by some motivation for the study of L-presentations. Recall Graham Higman's Embedding Theorem.

Theorem 2.3 ([Hig61] or [LS70, Section IV.7]). *A countable group G can be embedded in a finitely presented group $\widehat{G}$ if and only if it is recursively presented.*

The first proof by Higman was unconstructive; since then, explicit constructions of $\widehat{G}$ were given [Aan73,AC80,OS01]. They require, however, a good mastery of Turing- or S-machine programming. I am not aware of an explicit finitely presented group containing $\mathbb{Q}$. In contrast, a finitely L-presented group containing $\mathbb{Q}$ follows:

$$\mathbb{Q} = \left\langle x_1, x_2, \ldots \,\middle|\, x_n x_{n+1}^{-n-1} \,\forall n \geqslant 1\right\rangle,$$

where x_n should be interpreted as $1/n! \in \mathbb{Q}$. We embed $\mathbb{Q}$ in the finitely generated group

$$G = \langle x, y \mid x^n y^n (x^{n+1} y^{n+1})^{-n-1}\ \forall n \geqslant 1\rangle = \langle x, y \mid yx(x^{n+1}y^{n+1})^n\ \forall n \geqslant 1\rangle$$

through $x_n \mapsto x^n y^n$; now G embeds in the finitely L-presented group H given by

$$H = \langle x, y, a, b, c, d, e, c', d', e' \mid c^{-1}c', d^{-1}d', e^{-1}e', [\langle d, e\rangle, \langle x, y\rangle] \mid \phi_1, \phi_2, \phi_3, \phi_4 \mid yxbac, (e')^{d'} c'\rangle,$$

where

$$\phi_1 : \{a \mapsto bac, \qquad \phi_2 : \begin{cases} a \mapsto 1, \\ b \mapsto d^{-1}ybxd, \end{cases} \qquad \phi_3 : \begin{cases} a \mapsto 1, \\ b \mapsto yex, \end{cases} \qquad \phi_4 : \begin{cases} c' \mapsto c'c, \\ d' \mapsto d'd, \\ e' \mapsto e'e, \end{cases}$$

it being understood that unspecified generators map to themselves. Indeed, $n-1$ applications of ϕ_1 to $yxbac$ yield $yxb^n ac^n$; then n applications of ϕ_2 yield $yx((d^{-1}y)^n b(xd)^n)^n c^n$; then ϕ_3 and the commutation relations yield $yx(y^{n+1}x^{n+1})^n d^{-n}e^n d^n c^n$. On the other hand, $\phi_4^{n-1}((d')^{-1}e'd'c')$ yields $d^{-n}e^n d^n c^n$, so $yx(y^{n+1}x^{n+1})^n = 1$ in H, whence $x^n y^n = (x^{n+1}y^{n+1})^{n+1}$ in H. Any other sequence of operations ϕ_i would give a long relation containing non-$\{x, y\}$ symbols, so G embeds in H.

Some finitely L-presented groups embed nicely in finitely presented groups; recall that the HNN extension $\Omega(G, H \xrightarrow{\phi} K)$ is *ascending* if $H = G$.

Theorem 2.4. *Let G be finitely L-presented by an injective ascending L-presentation. Then a finitely presented group $\widehat{G}$ containing G can be effectively constructed. Moreover, $\widehat{G}$ is an ascending HNN extension of G by a finite number of stable letters.*

In case G is amenable, $\widehat{G}$ is a finitely presented amenable group containing G.

Proof. Let $\langle S \| \Phi \mid R\rangle$ be a finite ascending L-presentation of G. Consider the group

$$\widehat{G} = \langle S \cup \Phi \mid R \cup \{s^\phi = \phi(s)\}_{s \in S, \phi \in \Phi}\rangle.$$

It is finitely presented, and the map $G \to \widehat{G}$ defined by sending $s \in S$ to s is a well-defined injective homomorphism, since the $\phi : S^* \to S^*$ induce injective homomorphisms of G. □

Note that if G's presentation is not injective, then G needs not embed in $\widehat{G}$ anymore; for example, consider

$$G = \langle x, y \mid \mid \phi \mid x^{2y-3}\rangle,$$

with $\phi(x) = x^2$ and $\phi(y) = y$. This is the non-Hopfian Baumslag–Solitar group $B(2, 3)$, containing non-Abelian free subgroups. Inside the group $\widehat{G}$, consider the subgroup generated by x and y. It is isomorphic to $\mathbb{Z}[\frac{1}{6}] \rtimes \langle \frac{3}{2}\rangle$, and hence a strict quotient of G.

As a consequence of Theorem 2.4, we may construct finitely generated subgroups of hyperbolic groups that are not hyperbolic. Recall that for a fixed set W of words closed under cyclic permutation, a *piece* is a common subword of two distinct words of W. We say W satisfies the *small cancellation condition* $C'(\epsilon)$ if every piece p in $w \in W$ satisfies $|p| < \epsilon|w|$, and we say W satisfies $C(n)$ if no $w \in W$ can be factored in n pieces or fewer.

Ilya Kapovich and Dani Wise give in [KW01] a sufficient condition for a finitely L-presented group to have small cancellation, and in this way constructed a hyperbolic group with a non-co-Hopfian one-ended two-generator subgroup.

Thus there exists a $C'(\frac{1}{6})$ group (which is therefore word-hyperbolic)

$$\widehat{G} = \left\langle x, y, \tau \,\middle|\, r(x, y), x^{\tau-\phi(x)}, y^{\tau-\phi(y)} \right\rangle$$

which is an HNN extension of the group $G = \langle x, y \| \phi | r\rangle$, and such that the infinite presentation $G = \langle x, y | \bigcup_{n\geqslant 0} \phi^n(r)\rangle$ satisfies the small cancellation $C(6)$ condition. Therefore G is not finitely presentable, and hence not word-hyperbolic, and not quasi-convex in G. This is among the simplest examples of hyperbolic groups with non-hyperbolic finitely generated subgroups. For concreteness, consider the group

$$G = \left\langle x, y \middle| \middle| \phi \middle| (xy)^7 \right\rangle,$$

with $\phi(x) = x^7$ and $\phi(y) = y^7$, embedding in

$$\widehat{G} = \left\langle x, y, \phi \,\middle|\, x^{\phi-7}, y^{\phi-7}, (xy)^7 \right\rangle.$$

Proposition 2.5. *If G is finitely presented, then it is finitely L-presented. There are non-finitely L-presented groups, and there are finitely L-presented, but not finitely presented, groups.*

Proof. The first claim is obvious: finite L-presentations with $R = \Phi = \emptyset$ are precisely finite presentations.

There are only countably many finite L-presentations, but uncountably many finitely-generated groups, so "most" groups are not finitely L-presented.

Finally, Theorem 4.1 shows that the "lamplighter group" described there is finitely L-presented, but not finitely presented. □

Note, however, that it is not trivial to explicitly point at a non-finitely L-presentable group. A group having a non-recursively-enumerable presentation satisfying some small cancellation condition would be an example. The free group in a variety defined by infinitely many identities (they exist by [Ol'70]) is another one. More examples appear in the course of Lemma 2.8.

Proposition 2.6. *If G, H are finitely L-presented groups, then $G * H$ is finitely L-presented.*

If G is finitely L-presented and H, K are isomorphic finitely generated subgroups of G, then the HNN extension $\Omega(G, H \xrightarrow{\psi} K)$ is finitely L-presented.

Proof. Let $\langle S|\,Q|\,\Phi|\,R\rangle$ be a finite L-presentation of G, and let $\langle T\,|\,P|\,\Psi|\,U\rangle$ be a finite L-presentation of H. A finite L-presentation of $G * H$ is

$$\langle S \cup T\,|\,Q \cup P|\,\Phi \cup \Psi|\,R \cup U\rangle,$$

where it is understood that each $\phi \in \Phi$ is extended to a homomorphism $\phi : (S \cup T)^* \to (S \cup T)^*$ by mapping each $t \in T$ to itself; and similarly for each $\psi \in \Psi$.

Let now H be the subgroup of G generated by $T \subset S^*$. A presentation for the HNN extension of G by $\psi : H \to K$ is

$$\left\langle S \cup \{\psi\} \,\middle|\, Q \cup \left\{\psi(t)^{-1} t^{\psi}\right\}_{t \in T} \,\middle|\, \Phi \,\middle|\, R\right\rangle. \qquad \square$$

Proposition 2.7. *If G, H are finitely L-presented groups, then any split extension of G by H is finitely L-presented. If H is finitely presented, then any extension of G by H is finitely L-presented.*

Proof. Let $\langle S|\,Q|\,\Phi|\,R\rangle$ be a finite L-presentation of G; let $\langle T\,|\,P|\,\Psi|\,U\rangle$ be a finite L-presentation of H; let X be an extension of G by H, given as $1 \to G \to X \to H \to 1$. Let σ be a section of H to X; in case the extension splits, we suppose that σ is a group homomorphism.

Each relator $p \in P$ lifts through σ to an element $g_p \in G$, so we may define $P' = \{p g_p^{-1} \mid p \in P\}$, a set of relators in X. Since G is normal in X, we also have $s^{\sigma(t)} = g_{s,t} \in G$ for each $s \in S$, $t \in T$. Consider now the presentation

$$\left\langle S \cup T \,\middle|\, Q \cup P' \cup \left\{s^t g_{s,t}^{-1}\right\}_{s \in S, t \in T} \,\middle|\, \Phi \cup \Psi \,\middle|\, R \cup U\right\rangle, \tag{1}$$

where it is understood that each $\phi \in \Phi$ is extended to a homomorphism $\phi : (S \cup T)^* \to (S \cup T)^*$ by mapping each $t \in T$ to itself; and similarly for each $\psi \in \Psi$.

If X is a split extension, then $g_p = 1$ for all $p \in P$, and similarly all $\phi(u)$ (with $u \in U$ and $\phi \in \Phi^*$) are relations in X. If H is finitely presented, we may suppose $U = \emptyset$ and again all relations given in (1) are satisfied.

We have shown that in the cases considered X is a quotient of (1). Let now w be a word in $S \cup T$ equal to 1 in X. The relations $s^t = g_{s,t}$ allow w to be written as $s_1 \cdots s_n t_1 \cdots t_m$; then projecting to H gives $t_1 \cdots t_m = 1$ by applying relations in H. The same relations in (1) will reduce $s_1 \cdots s_n t_1 \cdots t_m$ to a word in S^*; the corresponding element of G can be reduced to 1 using relations in G, and these same relations exist in X, so $w = 1 \in X$ and (1) is a presentation of X. $\square$

Note that there are extensions of finitely L-presented groups that are not finitely L-presented; more precisely,

Lemma 2.8. *There are uncountably many non-isomorphic extensions of $\mathbb{Z}/2$ by $\mathbb{Z}/2 \wr \mathbb{Z}$.*

As a consequence, there are uncountably many such extensions that are not finitely L-presented.

Proof. $H^2(\mathbb{Z}/2 \wr \mathbb{Z}, \mathbb{Z}/2) = (\mathbb{Z}/2)^\infty$—see Section 2.3. $\square$

Proposition 2.9. *If G is a finitely L-presented group, then any finite-index subgroup of G is finitely L-presented.*

If $N \lhd G$ is finitely generated as a normal subgroup of a finitely L-presented group G, then G/N is finitely L-presented.

Proof. Let $\langle S \mid Q \mid \Phi \mid R\rangle$ be a finite L-presentation of G, and let X be a right transversal of the finite-index subgroup H of G. In view of Proposition 2.7, we may suppose H is normal in G, since any finite-index subgroup is a finite extension of its core, which is normal of finite index.

We then have $G = \bigcup_{x\in X} Hx = \bigcup_{x\in X} xH$. For $g \in G$, let $\overline{g} \in X$ denote its coset representative. By the Reidemeister–Schreier method, H is generated by the finite set $T = \{s^x\}_{x\in X, s\in S}$, and a presentation of H is given by

$$\Big\langle T \Big| \big\{\widetilde{q^x}\big\}_{q\in Q, x\in X} \cup \big\{\widetilde{\phi(r)^x}\big\}_{\phi\in\Phi^*, r\in R, x\in X}\Big\rangle,$$

where $\widetilde{w}$ is a rewriting of w as a word over T. Now each $\phi \in \Phi$ induces naturally a monoid homomorphism $\widetilde{\phi}$ over T^*, and since $\widetilde{\phi(r)^x} = \widetilde{\phi}(\widetilde{r^x})$, a finite L-presentation for H is given by

$$\big\langle T \big| \big\{\widetilde{q^x}\big\}_{q\in Q, x\in X} \big| \big\{\widetilde{\phi}\big\}_{\phi\in\Phi} \big| \big\{\widetilde{r^x}\big\}_{r\in R, x\in X}\big\rangle.$$

For the second statement of the proposition, let $\langle S \mid Q \mid \Phi \mid R\rangle$ be a finite L-presentation of G and let T be a finite generating set for N. Then

$$\langle S \mid Q \cup N \mid \Phi \mid R\rangle$$

is a finite L-presentation of G/N. $\square$

Proposition 2.10. *If G, H are finitely L-presented groups, and either G is Abelian or H is finite, then the restricted wreath product $G \wr H$ is finitely L-presented.*

Problem 2.11. The corresponding assertion with "finitely L-presented" replaced by "finitely presented" does not hold. Under which conditions does the statement hold, for non-Abelian G and infinite H?

Proof. If H is finite, then $G \wr H$ is finitely L-presented by Proposition 2.7. Let us assume then that G is Abelian. Let $\langle S \mid Q \mid \Phi \mid R\rangle$ be a finite L-presentation of G, and let $\langle T \mid P \mid \Psi \mid U\rangle$ be a finite L-presentation of H. An L-presentation of $G \wr H$ is

$$\Big\langle S \cup T \Big| Q \cup P \cup \big\{\big[s_1, s_2^h\big]\big\}_{s_1, s_2\in S, h\in H} \Big| \Phi \cup \Psi \Big| R \cup U\Big\rangle,$$

where it is understood that each $\phi \in \Phi$ is extended to a homomorphism $\phi : (S \cup T)^* \to (S \cup T)^*$ by mapping each $t \in T$ to itself; and similarly for each $\psi \in \Psi$. This L-presentation is in general not finite, but this can be remedied by introducing new generators $\overline{S}$ in bijection with S and new homomorphisms Ω_T in bijection with T:

$$G \wr H = \left\langle S \cup T \cup \overline{S} \,\middle|\, Q \cup P \cup \left\{s^{-1}\bar{s}\right\}_{s \in S} \,\middle|\, \Phi \cup \Psi \cup \Omega_T \,\middle|\, R \cup U \cup \left\{[s_1, \bar{s}_2]\right\}_{s_1, s_2 \in S}\right\rangle,$$

where $\omega_t \in \Omega_T : (S \cup T \cup \overline{S})^* \to (S \cup T \cup \overline{S})^*$ is defined by $\omega_t(\bar{s}) = \bar{s}^t$ and $\omega_t(s) = s$ and $\omega_t(t') = t'$. Indeed, the new generators $\bar{s}$ do not enlarge G, since $\bar{s} = s$ is a relation; also,

$$\left[s_1, s_2^h\right] = \left[s_1, \overline{s_2}^h\right] = \left[s_1, \overline{s_2}^{t_1 \cdots t_n}\right] = \left[s_1, \omega_{t_1} \cdots \omega_{t_n}(\overline{s_2})\right] = \omega_{t_1} \cdots \omega_{t_n}\big([s_1, \overline{s_2}]\big) = 1$$

is a relation, for all $h = t_1 \cdots t_n \in H$. $\square$

Problem 2.12. Let G be a finitely L-presented group generated by S, let H be a finitely generated subgroup, and let X be a transversal of H in G which is a regular subset of S^*. Under which extra conditions is H finitely L-presented?

2.2. Identities

We now show that groups defined by finitely many identities are all finitely L-presented. Recall that an identity is a word $w \in F_Y$, and that the group G *satisfies the identity* w if $f(w) = 1$ for all $f : F_Y \to G$. For instance, all Abelian groups satisfy the identity $[y_1, y_2]$. The *free group on S with respect to w* is $F_S / \langle f(w)\ \forall f : F_Y \to F_S\rangle$. It is the largest group satisfying w, in the sense that every group generated by S and satisfying w is a quotient of it. These groups are also sometimes referred to as *relatively free groups* of finitely based varieties [Neu67]. In that spirit, a group has *presentation $\langle S | R\rangle$ within a variety* if it is the quotient of the free group on S in that variety by the normal closure of R.

Proposition 2.13. *Let G be finitely generated and finitely presented with respect to the identity w. Then G is finitely L-presented.*

Proof. It suffices to prove the claim for a relatively free group, since the quotient of a finitely L-presented group by a finitely normally generated normal subgroup remains finitely L-presented.

Let us then suppose G relatively free and generated by X, and write $w = w(y_1, \ldots, y_n) \in F_Y$. For $x \in X^{\pm 1}$ and $y \in Y$, define the endomorphism ϕ_{xy} of $F_{X \sqcup Y}$ by $\phi_{xy}(y) = xy$, and $\phi_{xy}(s) = s$ for all other $s \in X \sqcup Y$. Then the following is a finite L-presentation of G:

$$\left\langle X \sqcup Y \,\middle|\, Y \,\middle|\, \{\phi_{xy}\}_{x \in X^{\pm 1}, y \in Y} \,\middle|\, \{w\}\right\rangle.$$

Indeed, write $\Phi = \{\phi_{xy}\}_{x \in X^{\pm 1}, y \in Y}$. Then

$$\begin{aligned}\left\langle X \sqcup Y \,\middle|\, Y \,\middle|\, \Phi \,\middle|\, \{w\}\right\rangle &= \left\langle X \sqcup Y \,\middle|\, Y \cup \Phi(w)\right\rangle = \left\langle X \sqcup Y \,\middle|\, Y \cup \left\{w\big(w_1(X)y_1, \ldots, w_n(X)y_n\big)\right\}\right\rangle \\ &= \left\langle X \,\middle|\, \left\{w\big(w_1(X), \ldots, w_n(X)\big)\right\}\right\rangle = \left\langle X \,\middle|\, f(w)\ \forall f : F_Y \to F_X\right\rangle,\end{aligned}$$

where the $w_i(X)$ are arbitrary words over X, and $f: F_Y \to F_X$ is given by $f(y_i) = w_i(X)$. $\square$

As a consequence, the free Burnside groups (defined by the identity $a^n \in F_a$), the rank-r free solvable groups, etc., are finitely L-presented. Moreover:

Corollary 2.14. *Any finitely generated group in the variety $\mathfrak{A}\mathfrak{N}_k$ of Abelian-by-(nilpotent of degree k) groups is finitely L-presented.*

Proof. By [Hal54], every group in the variety $\mathfrak{A}\mathfrak{N}_k$ is the quotient of the free group in that variety (defined by the identity $[[x_1, \dots, x_k], [y_1, \dots, y_k]] \in F_{x_i, y_i}$) by a finite number of relations. $\square$

2.3. Schur multipliers

It is well-known, by Issai Schur and Heinz Hopf's formula [Bro94, Theorem 5.3], that the Schur multiplier $H_2(G, \mathbb{Z})$ ($= H^2(G, \mathbb{C}^\times)$ for finite groups) of a group G can be computed from a presentation of G; namely, given a presentation $G = \langle S | T \rangle$, we have

$$H_2(G, \mathbb{Z}) = \frac{\langle T \rangle^\# \cap [F_S, F_S]}{[\langle T \rangle^\#, F_S]}.$$

As a consequence, a finitely presented group necessarily has a finite-rank Schur multiplier. (Note, however, that the converse is not true—see Theorem 4.2.) We note that Hopf's formula extends to L-presentations.

Let us first recall a few facts on Schur multipliers; see [Kar87] for further details:

- Norman Blackburn's result [Bla72]

$$H_2(H \wr G, \mathbb{Z}) = H_2(G, \mathbb{Z}) \oplus H_2(H, \mathbb{Z}) \oplus \frac{\{f : G \to H/H' \otimes H/H'\}}{\{f(x^{-1}) = \tau f(x)\}}, \tag{2}$$

  where $\tau : H/H' \otimes H/H' \to H/H' \otimes H/H'$ sends $h \otimes h'$ to $h' \otimes h$, and the f above are just set maps.
- A special case of the Künneth formula,

$$H_2(G \times H, \mathbb{Z}) = H_2(G) \oplus H_2(H) \oplus \left(G/G' \otimes H/H'\right). \tag{3}$$

- Shapiro's lemma: for an exact sequence $1 \to N \to G \to Q \to 1$,

$$H_2(N, \mathbb{Z}) = H_2(G, \mathbb{Z}Q), \tag{4}$$

  with the G-action on $\mathbb{Z}Q$ induced by the quotient map $G \to Q$.

Theorem 2.15. *Let G admit a finite L-presentation $\langle S | Q | \Phi | R \rangle$. Then $H_2(G, \mathbb{Z}) = A \oplus \bigoplus_{\Phi^*} B$, where A and B are finitely-generated Abelian groups.*

Proof. Write $F = F_S$, and $W = \langle \Phi^*(R)\rangle^\#$. Consider the group $W/[W,F]$. It is Abelian and generated by $\Phi^*(R)$. The maps $\phi \in \Phi$ are such that $\operatorname{coker}\phi$ is finitely generated (by R), and we may filter R along Φ^*. For each $\phi \in \Phi$, write

$$1 \longrightarrow \ker\phi \longrightarrow W \xrightarrow{\phi} W \longrightarrow A_\phi \oplus B_\phi \longrightarrow 1,$$

where A_ϕ splits back into W and B_ϕ does not. Then W lies inside $\bigoplus_{\phi\in\Phi} B_\phi \oplus \bigoplus_{\phi\in\Phi^*} A_\phi$, so W is of the required form. The Schur multiplier is obtained from W by extending by $\langle S\rangle^\#/[F,S]$ (which has finite rank), and restricting to $[F,F]$, both operations preserving the claimed form of $H_2(G,\mathbb{Z})$. □

It follows, for instance, that $H_2(G,\mathbb{Z})$ may be neither $\mathbb{Q}$ nor $\mathbb{Z}[1/n]$, for a finitely L-presented group. However, the Schur multiplier may be trivial, as in Theorem 4.2, or of infinite rank, as in the examples of branch groups of Section 3.2.

3. Branch groups

The purpose of this section is to prove the following general results.

Theorem 3.1. *Let G be a finitely generated, contracting, semi-fractal, regular branch group. Then G is finitely L-presented.*

Theorem 3.2. *Let G be a finitely generated, contracting, semi-fractal, regular branch group. Then G is not finitely presented.*

Even though I am sure that the contracting hypothesis is not needed in Theorem 3.2, I have been unable to prove it without that extra condition.

We start by recalling some definitions from [BG00,Gri00] concerning branch groups. We fix an integer $d \geqslant 2$, and the finite alphabet $\Sigma = \mathbb{Z}/d\mathbb{Z}$, written $\{1,\dots,d\}$ for convenience. The *d-regular rooted tree* is the free monoid Σ^*. A tree automorphism $g \in \operatorname{Aut}\Sigma^*$ is a bijective map $g : \Sigma^* \to \Sigma^*$ that preserves prefixes, i.e., such that $g(\sigma\tau) \in g(\sigma)\Sigma^{|\tau|}$ for all $\sigma,\tau \in \Sigma^*$. There is an isomorphism between the subtree $\sigma\Sigma^*$ and Σ^*, given by left-cancellation of σ. It induces an isomorphism $\pi_\sigma : \operatorname{Aut}(\sigma\Sigma^*) \to \operatorname{Aut}(\Sigma^*)$.

A *d-rooted group* is a finitely generated subgroup G of $\operatorname{Aut}\Sigma^*$. The *rooted automorphism* is the automorphism $a \in \Sigma^*$ defined by

$$a(\sigma_1\sigma_2\cdots\sigma_n) = (\sigma_1+1)\sigma_2\cdots\sigma_n.$$

Fix a rooted group G, let $\operatorname{Stab}_G(\sigma)$ be the stabiliser in G of the vertex $\sigma \in \Sigma^*$, and set $\operatorname{Stab}_G(n) = \bigcap_{\sigma\in\Sigma^n} \operatorname{Stab}_G(\sigma)$. Restriction induces a map

$$\pi_\sigma : \operatorname{Stab}_G(\sigma) \to \operatorname{Aut}\bigl(\sigma\Sigma^*\bigr) \to \operatorname{Aut}\Sigma^*.$$

The group G is *fractal* if $\pi_\sigma \operatorname{Stab}_G(\sigma) = G$ for all $\sigma \in \Sigma^*$, and *semi-fractal* if $\pi_\sigma \operatorname{Stab}_G(\sigma) \leqslant G$ for all $\sigma \in \Sigma^*$. In that case, the map

$$\psi = (\psi_1, \dots, \psi_d) : \operatorname{Stab}_G(1) \to G^\Sigma$$

defined by $\psi_i(g) = \pi_i(g|_{i\Sigma^*})$ is an embedding. It extends to a map still written $\psi : G \to G \wr \mathfrak{S}_\Sigma$, by lifting ψ to G using the natural map $G \to \mathfrak{S}_\Sigma$ given by restriction to the first level of the tree.

The *rigid stabiliser* of the vertex σ is

$$\operatorname{Rist}_G(\sigma) = \bigcap_{\tau \notin \sigma\Sigma^*} \operatorname{Stab}_G(\tau)$$

and the *rigid level stabiliser* of level n is

$$\operatorname{Rist}_G(n) = \prod_{\sigma \in \Sigma^n} \operatorname{Rist}_G(\sigma).$$

Note $\operatorname{Rist}_G(\sigma) < \operatorname{Stab}_G(\sigma)$ and $\operatorname{Rist}_G(n) < G$ for all $\sigma \in \Sigma^*$ and $n \in \mathbb{N}$.

The group G is *level-transitive* if it acts transitively on Σ^n for all $n \in \mathbb{N}$. In that case, $\operatorname{Stab}_G(\sigma)$ and $\operatorname{Rist}_G(\sigma)$ depend, up to isomorphism, only on the length of σ.

Definition 3.3. The group G is a *branch* group if it is level-transitive, and $\operatorname{Rist}_G(n)$ has finite index in G for all n. It is *weak branch* if all $\operatorname{Rist}_G(\sigma)$ are non-trivial (and hence infinite). It is *regular branch* if $[G : \pi_\sigma \operatorname{Rist}_G(\sigma)]$ is (finite and) constant for all long enough $\sigma \in \Sigma^*$. In that case, there is a finite-index subgroup $K \leqslant G$ such that $K^\Sigma \leqslant \psi(K)$, and G is *regular branch over* K.

Definition 3.4. Let G be a branch group generated by a finite set S, and consider the induced word metric on G. We say G is *contracting* if there is a constant D such that for every word $w \in S^*$ representing an element of $\operatorname{Stab}_G(1)$, writing $\psi(w) = (w_1, \dots, w_d)$, we have

$$|w_i| < |w| \quad \text{for all } i \in \Sigma, \quad \text{as soon as} \quad |w| > D. \tag{5}$$

It then follows that there is an algorithm $\mathcal{A}$ solving the word problem in G: in this algorithm, we only assume that given a group generator we know its action on the top level Σ of the tree, and that given a word representing an element of $\operatorname{Stab}_G(1)$ we may compute explicitly $\psi(w)$.

Initialization: Let $V \subset S^*$ be the set of all words of length at most D, and let $W \subset V$ be the set of words acting trivially on Σ. Note that ψ is a well-defined map from W to V^d. Assign to each $v \in V$ a flag, that is either "trivial," "non-trivial," or "unknown yet." Initially all flags are "unknown yet."

For each $v \in V$ flagged "unknown yet," if $v \in V \setminus W$ or $\psi(v)$ has a component flagged "non-trivial," flag v as "non-trivial." If $\psi(v)$ has all components flagged "trivial," flag v as "trivial." Repeat the above procedure until no more flags are changed. Then flag all "unknown yet" words as "trivial."

Computation: Let $w \in S^*$ be a word of which one asks whether it is trivial in G. If w belongs to V, its flag answers the word problem. If w acts non-trivially on Σ, it is non-trivial. Finally, if w acts trivially on Σ, write $\psi(w) = (w_1, \dots, w_n)$. By property (5), each w_i is strictly shorter than w, so the algorithm can be applied inductively to it. Then w is trivial if and only if all w_i are trivial.

Only one point deserves a special justification, and that is the flagging of "unknown yet" words as trivial. This is because such words act trivially on the tree, so belong to $\bigcap_{n \geqslant 0} \mathsf{Stab}_G(n)$, which by assumption is trivial.

3.1. Proof of Theorem 3.1

Lemma 3.5. *Let $G = \langle S \rangle$ be a finitely generated group and let $H = \langle T \rangle$ be a finite-index subgroup of G, for some $T \subset S^*$. Let $\widetilde{S}$ be a set in bijection with S, and for $w = s_1 \cdots s_n \in S^*$ set $\widetilde{w} = \tilde{s}_1 \cdots \tilde{s}_n \in \widetilde{S}^*$.*

There exists a finitely presented group $\Gamma = \langle \widetilde{S} \rangle$ such that $\pi : \Gamma \xrightarrow{\tilde{s} \mapsto s} G$ is an epimorphism, and $\pi^{-1}(H) = \langle \widetilde{T} \rangle$ in Γ.

Proof. Consider first F_S with its natural projection $\pi : F_S \to G$, and set $\Delta = \pi^{-1}(H)$. Since Δ has finite index in F_S, it is finitely generated, say by the set U. Our purpose is to find a quotient of F_S in which Δ is generated by T. For each $u \in U$, let w_u be an expression of $\pi(u)$ over T. It then suffices to consider

$$\Gamma = \langle S \mid u^{-1} w_u \ \forall u \in U \rangle. \qquad \square$$

In words, Γ a finitely presented group such that the subgroup lattice between G and $\langle T \rangle$ is isomorphic to the lattice between Γ and $\langle T \rangle$, where the different $\langle T \rangle$'s lie in different groups.

Proof of Theorem 3.1. Let G be regular branch on its subgroup K, and fix generating sets S for G and T for K. It loses no generality to assume $K \leqslant \mathsf{Stab}_G(1)$, since one may always replace K by $K \cap \mathsf{Stab}_G(1)$. Let Γ_0 be the group given by Lemma 3.5 for $H = K$. Let Δ_0 be the natural lift of $\mathsf{Stab}_G(1)$ to Γ_0; and let Υ_0 be the natural lift of K to Γ_0.

Let U be a generating set of $\mathsf{Stab}_1(G)$ (so Δ_0 is generated by $\widetilde{U}$), and let $\widetilde{\psi} : \Delta_0 \to \Gamma_0^d$ be the natural lift of $\psi : \mathsf{Stab}_G(1) \to G^d$; it maps $\tilde{u}$ to $\widetilde{\psi(u)}$, where the wide tilde is applied to all d factors of $\psi(u)$. Note that $\widetilde{\psi}$ satisfies the contracting condition for the same constant D as ψ.

Since G is regular branch, there is an embedding $1^i \times K \times 1^{d-1-i} \hookrightarrow K$, from which for each generator $t \in T$ of K we may choose a word $\phi_i(t) \in T^*$ such that $\psi(\phi_i(t)) = (1, \dots, t, \dots, 1)$ with the 't' in position i.

Now write $\widetilde{\psi}(\phi_i(t)) = (r_{t,1}, \dots, r_{t,i} t, \dots, r_{t,d})$ for some $r_{t,i} \in \Upsilon_0$. These elements' images in K are trivial, since $\widetilde{\psi}$ is a lift of ψ. Furthermore, since $\widetilde{\psi}$ is contracting, one may replace $\{r_{t,i}\}_{t\in T, i\in\Sigma}$ by its iterates under all $\pi_i\widetilde{\psi}$, where π_i is the projection on the ith factor, and still obtain a finite set of relations.

Let Γ be the quotient of Γ_0 by this set's normal closure. Then Γ is finitely presented and surjects onto G (since $r_{t,i} \cong 1$ in G). Let Δ and Υ be the images of Δ_0 and Υ_0 in Γ, and note that ψ lifts again to $\widetilde{\psi}$ on Γ, because the new relators r_{t_i} map to other new relators.

The data are summed up in the following diagram, which should be viewed as a "chair with ψ and $\widetilde{\psi}$ coming forward":

$$\begin{tikzcd}
 & \Gamma \arrow[r, two heads] \arrow[d, dash] & \Gamma \arrow[d, dash] & \\
\Gamma^d \arrow[d, dash] & \Delta \arrow[l, "\widetilde{\psi}"'] \arrow[r, two heads] \arrow[d, dash] & \mathsf{Stab}_G(1) \arrow[r, hook, "\psi"] \arrow[d, dash] & G^d \arrow[d, dash] \\
\Upsilon^d & \Upsilon \arrow[r, two heads] & K & K^d \arrow[l, hook']
\end{tikzcd}$$

Since $\operatorname{Im}\widetilde{\psi}$ contains Υ^d, it has finite index in Γ^d. Since Γ^d is finitely presented, $\operatorname{Im}\widetilde{\psi}$ too is finitely presented. Similarly, Δ is finitely presented, and we may express $\ker\widetilde{\psi}$ as the normal closure $\langle R_1\rangle^\#$ in Δ of those relators in $\operatorname{Im}\widetilde{\psi}$ that are not relators in Δ. Clearly, R_1 may be chosen to be finite.

We now use the assumption that G is contracting, with constant D. Let R_2 be the set of words over S of length at most D that represent the identity in G. Set $R = R_1 \cup R_2$, which clearly is finite.

We now consider T as a set distinct from S, and not as a subset of S^*. We extend each ϕ_i to a monoid homomorphism $(S\cup T)^* \to (S\cup T)^*$ by defining it arbitrarily on S.

Assume $\Gamma = \langle S \mid Q\rangle$, and let $w_t \in S^*$ be a representation of $t \in T$ as a word in S. We claim that the following is an L-presentation of G:

$$G = \bigl\langle S\cup T \bigm| Q \cup \bigl\{t^{-1}w_t\bigr\}_{t\in T} \bigm| \{\phi_i\}_{i\in\Sigma} \bigm| R_1\cup R_2\bigr\rangle. \tag{6}$$

For this purpose, consider the following subgroups Ξ_n of Γ: first $\Xi_0 = \{1\}$, and by induction

$$\Xi_{n+1} = \bigl\{\gamma\in\Delta \bigm| \widetilde{\psi}(\gamma)\in\Xi_n^d\bigr\}.$$

We computed $\Xi_1 = \langle R\rangle^\#$. Since G acts transitively on the nth level of the tree, a set of normal generators for Ξ_n is given by $\bigcup_{i\leqslant n}\phi^i(R)$, where ϕ is any choice of ϕ_i for $i\in\Sigma$. We also note that $\psi(\Xi_{n+1}) = \Xi_n^d$.

We will have proven the claim if we show $G = \Gamma\big/\bigcup_{n\geqslant 0}\Xi_n$. Let then $w\in\Gamma$ represent the identity in G. Applying to it $|w|$ times the map ψ, we obtain $d^{|w|}$ words that are all of length at most K, that is, that belong to Ξ_1. Then, since $\psi(\Xi_{n+1}) = \Xi_n^d$, we get $w\in\Xi_{|w|+1}$, and (6) is a presentation of G.

As a bonus, the presentation (6) expresses K as the subgroup of G generated by T. $\square$

A few remarks are in order. First, one can usually do with only one substitution, say ϕ_1, since in many cases the other ϕ_i are conjugates of ϕ_1. Second, ϕ_1 induces an isomorphism from K to its subgroup $K \times 1^{d-1}$, so there is a one-step HNN extension of G that is finitely presented—namely, the extension identifying K and $K \times 1^{d-1}$. Third, in many cases (but not all) ϕ_1 can be extended to an endomorphism of G; in that case, one may delete T from the generating set and obtain an ascending L-presentation.

In all cases, K admits an ascending L-presentation, so embeds in a finitely presented group L, and $\langle G, L\rangle$ is a finite extension of L, hence is a finitely presented group containing G.

Proof of Theorem 3.2. Since G is contracting, there is a constant D such that $|w_i| < |w|$ whenever $|w| > D$. This implies, using the triangular inequality, that there are constants $\eta < 1$ and D' such that $|w_i| < \eta|w|$ whenever $|w| > D'$.

Now levels can be "collapsed" in a branch group: for any k we may consider the (same) action of G on $(\Sigma^k)^*$, with map ψ given by k-fold composition of the original map ψ. The resulting group action is still branch.

However, the result of this process is that the constant η above can be replaced by any power of itself, say $1/2$, at the cost of enlarging the branching number of the tree.

The generating set can then be replaced by a ball of sufficiently large radius, so that the constant L becomes 1.

We have reached a "canonical situation," where the maps ψ and $\widetilde{\psi}$ satisfy $|w_i| \leqslant (1/2)(|w|+1)$ for all w.

Assume now by contradiction that G is finitely presented, say $G = \langle S \mid R\rangle$ with $\pi : F_S \to G$ the canonical map, and assume that the greatest length among the relators is minimal. All $r \in R$ being trivial in G, satisfy *a fortiori* $\pi(r) \in \mathsf{Stab}_G(1)$, so $\widetilde{\psi}(r) = (r_1, \dots, r_d)$ is well-defined. By the Reidemeister–Schreier process, a presentation of $G \times 1 \times \cdots \times 1$ is

$$\langle S \mid r_i \text{ for all } r \in R \text{ and } i \in \Sigma\rangle.$$

By our assumptions that $|r_i| \leqslant (1/2)(|r|+1)$ and $\max|r|$ is minimal, we must have $|r| \leqslant 1$ for all relations, so G is free. However, a free group may not contain commuting subgroups with trivial intersection, like $K \times 1 \times \cdots \times 1$ and $1 \times \cdots \times 1 \times K$. This is our required contradiction. □

3.2. Schur multipliers

In his paper [Gri99] Rostislav Grigorchuk computed explicitly the Schur multiplier $H_2(G,\mathbb{Z})$ of his group—he proved that $H_2(G) = (\mathbb{Z}/2)^\infty$. We outline here a general computation $H_2(G,\mathbb{Z})$ for branch groups G.

Theorem 3.6. *Let G be a finitely generated, contracting, semi-fractal, regular branch group. Then $H_2(G,\mathbb{Z}) \cong A \oplus B^\infty$, for finite Abelian groups A, B.*

As a consequence, all such groups are infinitely presented.

Proof. We concentrate on the exact sequence $1 \to K^d \to K \to Q \to 1$, for some finite group Q. By (4) and (3), $H_2(K, \mathbb{Z}Q) = H_2(K^d, \mathbb{Z}) = H_2(K, \mathbb{Z})^d \oplus (K/K' \otimes K/K')^{d(d-1)/2}$. Taking Q-invariants of the right-hand side collapses all d copies of $H_2(K, \mathbb{Z})$ together, but we are left with the equation $H_2(K, \mathbb{Z}) = H_2(K, \mathbb{Z}) \oplus B$, where B, a quotient of $(K/K' \otimes K/K')^{d(d-1)/2}$, is a finite group.

Then $H_2(G, \mathbb{Z})$ is obtained from $H_2(K, \mathbb{Z})$ by extension and quotient by finite-rank Abelian groups, and the claimed result follows. □

Note, as a corollary, that if K is perfect, then it is a finitely presented, infinitely related group with trivial Schur multiplier.

3.3. Perfect branch groups

We consider a class of branch groups, of special interest for being perfect. They form a subclass of the GGS groups studied in [BŠ01]. Let A be a finite, perfect, group acting transitively on Σ, with two elements $* \neq \dagger \in \Sigma$ such that $\mathsf{Stab}_A(*) \setminus \mathsf{Stab}_A(\dagger) \neq \emptyset$ (think, for instance, $\mathfrak{A}_5$).

Let $\overline{A}$ be a copy of A, and consider $\Gamma = A * \overline{A}$. Define an action of Γ on Σ^* by

$$(\sigma_1\sigma_2\cdots\sigma_n)^a = \sigma_1^a\sigma_2\cdots\sigma_n,$$
$$(\sigma_1\sigma_2\cdots\sigma_n)^{\bar{a}} = \begin{cases} \sigma_1\sigma_2^a\sigma_3\cdots\sigma_n & \text{if } \sigma_1 = *, \\ \sigma_1(\sigma_2\sigma_3\cdots\sigma_n)^{\bar{a}} & \text{if } \sigma_1 = \dagger, \\ \sigma_1\sigma_2\cdots\sigma_n & \text{otherwise.} \end{cases}$$

Let G be the group defined by this action, i.e., the quotient of Γ by the kernel of the action.

Proposition 3.7. *G is a perfect finitely generated regular branch group over itself.*

Proof. Clearly G is perfect, being generated by two perfect groups, and finitely generated, being generated by two finite groups.

Note now that $\mathsf{Stab}_G(1) = \overline{A}^G$. The map $\psi : G \to G \wr_\Sigma A$ is given by

$$\psi(a) = (1, \dots, 1)a, \qquad \psi(\bar{a}) = (a, 1, \dots, 1, \bar{a})1,$$

where in this last expression the 'a' is at position $*$ and the '$\bar{a}$' is at position $\dagger$. The conditions on A imply that it contains an element x moving $\dagger$ but not $*$. The computation $\psi[\bar{a}, \bar{b}^x] = ([a, b], 1, \dots, 1)$ shows that $\psi(G)$ contains $A \times 1 \cdots \times \cdots \times 1$, since A is perfect; then $\psi(G)$ contains too

$$(a, 1, \dots, 1)^{-1}\psi(\bar{a}) = \big(a^{-1}, 1, \dots, 1\big)(a, 1, \dots, 1, \bar{a}) = (1, \dots, 1, \bar{a}),$$

so $\psi(G)$ contains $1 \times \cdots \times 1 \times \overline{A}$, and since A is Y-transitive it contains $G \times \cdots \times G$. (Explicitly, we have $\psi^{-1}(G^\Sigma) = \mathsf{Stab}_G(1)$.) □

In this context, the statements of the previous section simplify: we have perfect regular branch groups G with $H_2(G,\mathbb{Z})=0$, that are finitely L-presented but infinitely presented. The group $\Gamma = A * \overline{A}$ is the same as the Γ of Theorem 3.1, and the subgroup Δ is $\mathop{\ast}_{a\in A}\overline{A}^a$. The map $\widetilde{\psi}$ is given by

$$\widetilde{\psi}\big(\bar{a}^b\big) = \big(a \text{ in position } *^b,\ \bar{a} \text{ in position } \dagger^b\big).$$

Let ϕ be some word substitution on Γ mapping g to $(g,1,\dots,1)$, as given by the computations in the previous theorem. We then have an L-presentation

$$G = \left\langle A, \overline{A} \;\middle|\; \;\middle|\; \phi \;\middle|\; \begin{array}{l} \bar{a}^{1-b} \text{ whenever } *^b = *, \dagger^b = \dagger, \\ \bar{a}^{1-b+c-d} \text{ whenever } *^b = *, \dagger^b = \dagger^c, *^c = *^d, \dagger^d = \dagger \text{ are all distinct}, \\ \bar{a}^{1-b+c-d(1-b+c)} \text{ whenever } *^b = \dagger^c = \dagger^d = *, \dagger^b = *^c, *^d = \dagger \\ \qquad\qquad\qquad\qquad \text{are all distinct}, \\ [\phi(a), \phi(b)^c] \text{ whenever } *^c \neq * \end{array} \right\rangle.$$

Indeed, the first three relations identify all products $\bar{a}^{b_1+\dots+b_n}$ with same ψ-image, and the last ones are the commutation relations lifted from $\Gamma \times \dots \times \Gamma$.

4. Examples

We start by an example of wreath product:

Theorem 4.1. *The following is an L-presentation of the "lamplighter group" $G = \mathbb{Z}/2 \wr \mathbb{Z}$:*

$$G = \big\langle a,b,t \,\big|\, a^2, a^{-1}b \,\big|\, \phi \,\big|\, [a,b] \big\rangle,$$

where $\phi : \{a,b,t\}^* \to \{a,b,t\}^*$ is given by

$$\phi(a) = a, \qquad \phi(b) = b^t, \qquad \phi(t) = t.$$

However, this group admits no finite presentation.

Proof. A presentation of G is

$$\big\langle a,t \,\big|\, a^2, \big[a, a^{t^i}\big] \forall i \in \mathbb{Z} \big\rangle.$$

By conjugating the last relation by t^i, we may assume the set of relators is a^2 and the $[a, a^{t^i}]$ with $i \geqslant 0$. The latter are precisely the relators obtained from $\phi^i([a,b])$ by applying the relation $a = b$.

It follows from [Bau61] that G is not finitely presented. Even better, (2) gives us $H_2(\mathbb{Z}/2 \wr \mathbb{Z}, \mathbb{Z}) = (\mathbb{Z}/2)^\infty$. □

Note, however, the following seemingly similar example, due to Gilbert Baumslag, which is finitely L-presented by arguments similar to those in Theorem 4.1:

Theorem 4.2 [Bau71]. *The group*

$$G = \left\langle a, b, t \,\middle|\, a^{t-4}, b^{2t-1}, \left[a, b^{t^i}\right] \forall i \in \mathbb{Z} \right\rangle$$

is an infinitely-presented metabelian group, with $H_2(G, \mathbb{Z}) = 0$.

This example was devised to show that the Schur multiplier's rank may be much smaller than the number of relators. In that view, we may ask the following question:

Problem 4.3. Do there exist non-finitely-L-presented groups with trivial Schur multiplier?

An interesting example of group acting on a rooted tree is the "Brunner–Sidki–Vieira group;" we rephrase their result in terms of L-presentations:

Theorem 4.4 [BSV99, Proposition 15]. *Consider the group* $G = \langle \mu, \tau \rangle$ *acting on* $\{1, 2\}^*$, *with* $\psi(a^{-1}\mu) = (1, \mu^{-1})$ *and* $\psi(a^{-1}\tau) = (1, \tau)$ (*so* τ *and* μ *act like* a *on the top node of the tree. Note that* G *is neither rooted nor branch, though* "*it is weak branch.*" *Writing* $\lambda = \tau\mu^{-1}$, G *admits the ascending* L*-presentation*

$$G = \left\langle \lambda, \tau \,\middle|\, \middle|\, \phi \,\middle|\, \left[\lambda, \lambda^{\tau}\right], \left[\lambda, \lambda^{\tau^3}\right] \right\rangle,$$

where ϕ *is defined by* $\tau \mapsto \tau^2$ *and* $\lambda \mapsto \tau^2\lambda^{-1}\tau^2$.

We may even conclude that $H_2(G, \mathbb{Z}) = (\mathbb{Z} \times \mathbb{Z})^\infty$, freely generated by the images of $\phi^n[\lambda, \lambda^\tau]$ and $\phi^n[\lambda, \lambda^{\tau^3}]$.

We now give presentations for four of the five "testbed groups" studied in [BG00, BG02].

4.1. *An L-presentation for G*

The group G, the *first Grigorchuk group*, is the 2-rooted group $G = \langle a, b, c, d \rangle$, with a the rooted element and b, c, d defined by

$$\psi(b) = (a, c), \qquad \psi(c) = (a, d), \qquad \psi(d) = (1, b).$$

G is a regular branch group over $K = \langle (ab)^2 \rangle^\#$.

Theorem 4.5. *The Grigorchuk group* G *admits the ascending* L*-presentation*

$$G = \left\langle a, c, d \,\middle|\, \middle|\, \sigma \,\middle|\, a^2, \left[d, d^a\right], \left[d^{ac}, \left(d^{ac}\right)^a\right] \right\rangle,$$

where $\sigma : \{a, c, d\}^* \to \{a, c, d\}^*$ *is defined by*

$$\sigma(a) = aca, \qquad \sigma(c) = cd, \qquad \sigma(d) = c.$$

Proof. Rephrasing of [Lys85]. □

4.2. An L-presentation for $\widetilde{G}$

The group $\widetilde{G}$, the *Grigorchuk supergroup*, is the 2-rooted group $G = \langle a, \tilde{b}, \tilde{c}, \tilde{d}\rangle$, with a the rooted element and $\tilde{b}, \tilde{c}, \tilde{d}$ defined by

$$\psi(\tilde{b}) = (a, \tilde{c}), \qquad \psi(\tilde{c}) = (1, \tilde{d}), \qquad \psi(\tilde{d}) = (1, \tilde{b}).$$

$\widetilde{G}$ is a regular branch group over $\widetilde{K} = \langle (a\tilde{b})^2, (a\tilde{d})^2\rangle^{\#}$. It is named thus because it contains G as a subgroup.

Theorem 4.6. *The group $\widetilde{G}$ admits the ascending L-presentation*

$$\widetilde{G} = \bigl\langle a, \tilde{b}, \tilde{c}, \tilde{d} \bigm| \bigm| \tilde{\sigma} \bigm| a^2, [\tilde{b}, \tilde{c}], [\tilde{c}, \tilde{c}^a], [\tilde{c}, \tilde{d}^a], [\tilde{d}, \tilde{d}^a], [\tilde{c}^{a\tilde{b}}, (\tilde{c}^{a\tilde{b}})^a], \\ [\tilde{c}^{a\tilde{b}}, (\tilde{d}^{a\tilde{b}})^a], [\tilde{d}^{a\tilde{b}}, (\tilde{d}^{a\tilde{b}})^a]\bigr\rangle,$$

where $\tilde{\sigma} : \{a, \tilde{b}, \tilde{c}, \tilde{d}\}^* \to \{a, \tilde{b}, \tilde{c}, \tilde{d}\}^*$ is defined by

$$a \mapsto a\tilde{b}a, \qquad \tilde{b} \mapsto \tilde{d}, \qquad \tilde{c} \mapsto \tilde{b}, \qquad \tilde{d} \mapsto \tilde{c}.$$

Proof. Rephrasing of [BG02, Proposition 5.6]. □

4.3. An L-presentation for Γ

The group Γ, the *Fabrykowski–Gupta group*, is the 3-rooted group $G = \langle a, r\rangle$, with a the rooted element and r defined by

$$\psi(r) = (a, 1, r).$$

Γ is a regular branch group over $\Gamma' = \langle [a, r]\rangle^{\#}$.

Theorem 4.7. *The Fabrykowski–Gupta group Γ admits the ascending L-presentation*

$$\bigl\langle a, r \bigm\| \sigma, \chi_1, \chi_2 \bigm| a^3, [r^{1+a^{-1}-1+a+1}, a], [a^{-1}, r^{1+a+a^{-1}}][r^{a+1+a^{-1}}, a]\bigr\rangle,$$

where $\sigma, \chi_1, \chi_2 : \{a, r\}^* \to \{a, r\}^*$ are given by

$$\begin{aligned} \sigma(a) &= r^{a^{-1}}, & \sigma(r) &= r, \\ \chi_1(a) &= a, & \chi_1(r) &= r^{-1}, \\ \chi_2(a) &= a^{-1}, & \chi_2(r) &= r. \end{aligned}$$

Proof. We follow Theorem 3.1. Consider first the group $F = \langle a, r \mid a^3, r^3\rangle$. Clearly, $F/F' \cong (\mathbb{Z}/3)^2 \cong \Gamma/\Gamma'$. Using the computer algebra program GAP, we compute a presentation for $\operatorname{Im}\widetilde{\psi}$, and rewrite its relators as words in X, where X is a generating set for Γ'. We also construct a group homomorphism $\sigma_0 : \Gamma' \to 1 \times 1 \times \Gamma'$. Then Theorem 3.1 gives a finite L-presentation for Γ with generators $\{a, r\} \cup X$.

We now note that σ_0 can be extended to a homomorphism $\sigma : \Gamma \to A \times R \times \Gamma$, where $A = \langle a \rangle$ and $R = \langle r \rangle$ have order 3. The substitution σ can be used instead of σ_0, giving rise to a simpler presentation with generators a, r.

Finally, we note that the presentation can be simplified from 6 iterated relators to 2 by introducing two extra substitutions χ_1, χ_2 induced by group automorphisms. $\square$

Note that for G and $\widetilde{G}$ the iterated relations are of the form $[x, x^a]$ where x belongs to a first-level rigid stabiliser. For Γ, however, one obtains fewer relations by considering more general expressions, as above.

4.4. An L-presentation for $\overline{\Gamma}$

The group $\overline{\Gamma}$, introduced in [BG00], is the 3-rooted group $G = \langle a, s \rangle$, with a the rooted element and s defined by

$$\psi(s) = (a, a, s).$$

Set $x = ta^{-1}$, $y = a^{-1}t$, and $K = \langle x, y \rangle$, a torsion-free index-3 subgroup of $\overline{\Gamma}$.

Theorem 4.8. *The groups K and $\overline{\Gamma}$ are not branch, but are finitely L-presented.*

Proof. We start by computing an L-presentation for K. As above, $\psi(K')$ contains $K' \times K' \times K'$; but $K/K' \cong \mathbb{Z}^2$ and neither $\overline{\Gamma}$ nor K are branch.

First, we chose generators of $\mathsf{Stab}_K(1)$:

$$\alpha = x^{-1}y = \bigl(x, 1, x^{-1}\bigr), \qquad \beta = y^{-1}x^{-1}y^{-1} = \bigl(y, 1, y^{-1}\bigr),$$
$$\gamma = x^{-1}y^{-1}x^{-1} = \bigl(1, x, x^{-1}\bigr), \qquad \delta = xy^{-1} = \bigl(1, y, y^{-1}\bigr).$$

Then, we chose generators of K':

$$\begin{aligned} e &= \beta^{-1}\delta\gamma = [y, xyx] = \bigl(y^{-1}, yx, x^{-1}\bigr),\\ f &= \gamma\beta^{-1}\delta = \bigl[x^{-1}y^{-1}x^{-1}, y\bigr] = \bigl(y^{-1}, xy, x^{-1}\bigr),\\ g &= \gamma^{-1}\alpha\beta = [x, y] = \bigl(xy, x^{-1}, y^{-1}\bigr),\\ h &= \beta\gamma^{-1}\alpha = \bigl[y^{-1}x^{-1}, y^{-1}\bigr] = \bigl(yx, x^{-1}, y^{-1}\bigr). \end{aligned}$$

The fact that K' is normal can be seen in the following conjugation relations:

$\swarrow$	e	f	g	h
x	$g^{-1}hf^{-1}g^{-1}$	$f^{-1}g^{-1}$	e	$g^{-1}he$
x^{-1}	ehf^{-1}	$h^{-1}e^{-1}$	$h^{-1}f^{-1}$	
y	$g^{-1}e^{-1}$	$h^{-1}e^{-1}$	$g^{-1}eh$	f
y^{-1}	$e^{-1}gf$	h	$f^{-1}g^{-1}$	$f^{-1}g^{-1}ef$

Define now the group L by its generators $S = \{x^{\pm 1}, y^{\pm 1}\}$ and relators $k^s = w_{k,s}$ for $s \in S$ and $k \in \{e, f, g, h\}$, where $w_{k,s}$ is the word in the above table. Note then that L' is generated by the words e, f, g, h.

As in the proof of Theorem 3.1, consider the map $\widetilde{\psi} : \mathsf{Stab}_L(1) \to L^3$ corresponding to $\psi : \mathsf{Stab}_K(1) \to K^3$, and by the Reidemeister–Schreier method compute a presentation for $\widetilde{\psi}(\mathsf{Stab}_L(1))$.

Assume $L = \langle S \mid Q \rangle$. A presentation for L^3 is $\langle S_1 \cup S_2 \cup S_3 \mid Q_1 \cup Q_2 \cup Q_3 \cup [S_i, S_{j \neq i}] \rangle$. The image of $\widetilde{\psi}$ can be described as $\{(u, v, w) \in L^3 \mid uvw \in L'\}$. We choose $\{x_3^m y_3^n\}_{m,n \in \mathbb{Z}}$ as Schreier transversal for this subgroup, and denote the Schreier generators

$$u_{imn} = x_3^m y_3^n x_i y_3^{-n} x_3^{-m-1}, \qquad v_{imn} = x_3^m y_3^n y_i y_3^{-n-1} x_3^{-m}.$$

Then these generators satisfy

$$\begin{aligned} &u_{1mn} = u_{100} = \alpha, && v_{1mn} = u_{200}^{-m} v_{100} u_{200}^m = \gamma^{-m} \beta \gamma^m, \\ &u_{2mn} = u_{200} = \gamma, && v_{2mn} = u_{100}^{-m} v_{200} u_{100}^m = \alpha^{-m} \delta \alpha^m, \\ &u_{3mn} = u_{200}^{-m} v_{100}^{-n} u_{200}^{-1} v_{100}^n u_{200}^{m+1}, && v_{3mn} = 1. \end{aligned}$$

The relators we obtain, in terms of $\alpha, \beta, \gamma, \delta$, are

$$\begin{gathered} [\alpha, \gamma],\ [\alpha\beta, \gamma\delta],\ \left[\alpha\gamma^{-1}, \beta^{-n}\gamma^{-1}\beta^n\right],\ \left[\gamma^{-n}\beta\gamma^n, \alpha^{-n}\delta\alpha^n\right] \quad \text{for all } n \in \mathbb{Z}, \\ w\left(\alpha\gamma^{-1}, \beta\right),\ w\left(\alpha^{-1}\gamma, \delta\right),\ w\left(\alpha^{-1}, \delta^{-1}\right) \quad \text{for all } w \in Q. \end{gathered}$$

This clearly gives a finite L-presentation for $\widetilde{\psi}(\mathsf{Stab}_L(1))$—compare with the proof of Theorem 4.1. Now the computation of a presentation for K can be finished as in the proof of Theorem 3.1.

Finally, a finite L-presentation for G can be obtained using Proposition 2.7. □

4.5. An L-presentation for $\overline{\overline{\overline{\Gamma}}}$

The group $\overline{\overline{\overline{\Gamma}}}$, the *Gupta–Sidki group*, is the 3-rooted group $G = \langle a, t \rangle$, with a the rooted element and t defined by

$$\psi(t) = \left(a, a^{-1}, t\right).$$

$\overline{\overline{\overline{\Gamma}}}$ is a regular branch group over $\overline{\overline{\overline{\Gamma}}}' = \langle [a, t] \rangle^{\#}$.

Theorem 4.9. *The Gupta–Sidki group $\overline{\overline{\overline{\Gamma}}}$ admits the L-presentation*

$$\left\langle a, t, u, v \,\middle|\, a^3, t^3, u^{-1}t^a, v^{-1}t^{a^{-1}} \,\middle|\, \sigma, \chi \,\middle|\, (tuv)^3, [v, t]\left[vt, u^{-1}tv^{-1}u\right], [t, u]^3[u, v]^3[t, v]^3 \right\rangle,$$

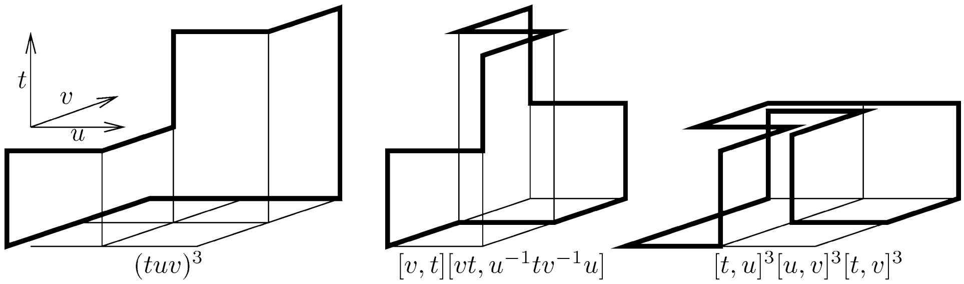

Fig. 1.

where $\sigma, \chi : \{t,u,v\}^* \to \{t,u,v\}^*$ *are given by*

$$\sigma : \begin{cases} t \mapsto t, \\ u \mapsto [u^{-1}t^{-1}, t^{-1}v^{-1}]t = u^{-1}tv^{-1}tuvt^{-1}, \\ v \mapsto t[tv,ut] = t^{-1}vutv^{-1}tu^{-1}, \end{cases} \qquad \chi : \begin{cases} t \mapsto t^{-1}, \\ u \mapsto u^{-1}, \\ v \mapsto v^{-1}. \end{cases}$$

Note that χ is induced by the automorphism of $\overline{\overline{\Gamma}}$ defined by $a \mapsto a$, $t \mapsto t^{-1}$; however, σ does not extend to an endomorphism of $\overline{\overline{\Gamma}}$.

Note also that all the iterated relators can be expressed as words over $\{t,u,v\}$ with 0-sum in each variable. Their most natural representation (see Fig. 1) is as closed paths in the $\{t,u,v\}$-space.

Then the fact that these elements are non-trivial relations translates to the fact that their projection on any plane $t=-u$, $u=-v$, or $v=-t$ gives a trivial path (up to $t^3=u^3=v^3=1$), while they themselves are not trivial paths. Incidentally, these projections are none but the $\psi_i : \langle t,u,v\rangle \to \langle a,t\rangle$, for $i \in \Sigma$.

Proof. We follow Theorem 3.1. Consider first the group $F = \langle a,t \mid a^3, t^3\rangle$. Clearly, $F/F' \cong (\mathbb{Z}/3)^2 \cong \overline{\overline{\Gamma}}/\overline{\overline{\Gamma}}'$. Using the computer algebra program GAP, we compute a presentation for $\operatorname{Im}\widetilde{\psi}$, and rewrite its relators as words in X, where X is a generating set for $\overline{\overline{\Gamma}}'$. We also construct a group homomorphism $\sigma_0 : \overline{\overline{\Gamma}}' \to 1 \times 1 \times \overline{\overline{\Gamma}}'$. Then Theorem 3.1 gives a finite L-presentation for $\overline{\overline{\Gamma}}$ with generators $\{a,t\} \cup X$.

We now note that σ_0 can be extended to a homomorphism $\sigma : \mathsf{Stab}_{\overline{\overline{\Gamma}}}(1) \to A \times A \times \overline{\overline{\Gamma}}$, where $A = \langle a\rangle$ has order 3. The substitution σ can be used instead of σ_0, giving rise to a simpler presentation with generators a,t,u,v, where t, $u = t^a$, $v = t^{a^{-1}}$ is a generating set for $\mathsf{Stab}_{\overline{\overline{\Gamma}}}(1)$.

Finally, we note that the presentation can be simplified from 6 iterated relators to 3 by introducing an extra substitution χ, induced by a group automorphism. □

Problem 4.10. Does there exist a finite ascending L-presentation for $\overline{\overline{\Gamma}}$?

In these examples, easy computations yield $H_2(G,\mathbb{Z}) = H_2(\widetilde{G},\mathbb{Z}) = (\mathbb{Z}/2)^\infty$ and $H_2(\Gamma,\mathbb{Z}) = H_2(\overline{\overline{\Gamma}},\mathbb{Z}) = (\mathbb{Z}/3)^\infty$.

Acknowledgments

The author acknowledges support from the "Swiss Mathematical Society," the Hebrew University of Jerusalem, and the University of California at Berkeley.

I thank Zoran Šunić, Denis Osin, Ilya Kapovich, Rostislav Grigorchuk, and Gulnara Arzhantseva for their entertaining discussions and careful reading of the text. Gulnara and Ilya generously offered contributions to Section 2.1, Denis to Section 2.2, and Zoran to Section 3.3.

Many results were obtained using the software system GAP [S$^+$93], whom I thank for its patient and silent permutation-grinding.

References

[Aan73] S. Aanderaa, A proof of Higman's embedding theorem using Britton extensions of groups, in: Word Problems: Decision Problems and the Burnside Problem in Group Theory, Conf. on Decision Problems in Group Theory dedicated to Hanna Neumann, Univ. California Irvine, CA, 1969, in: Stud. Logic Found. Math., Vol. 71, North-Holland, Amsterdam, 1973, pp. 1–18.

[AC80] S. Aanderaa, D.E. Cohen, Modular machines and the Higman–Clapham–Valiev embedding theorem, in: Word Problems, II, Conf. on Decision Problems in Algebra, Oxford, 1976, North-Holland, Amsterdam, 1980, pp. 17–28.

[Bau61] G. Baumslag, Wreath products and finitely presented groups, Math. Z. 75 (1960/1961) 22–28.

[Bau71] G. Baumslag, A finitely generated, infinitely related group with trivial multiplicator, Bull. Austral. Math. Soc. 5 (1971) 131–136.

[BG00] L. Bartholdi, R.I. Grigorchuk, Spectra of non-commutative dynamical systems and graphs related to fractal groups, C. R. Acad. Sci. Paris Sér. I Math. 331 (6) (2000) 429–434.

[BG02] L. Bartholdi, R.I. Grigorchuk, On parabolic subgroups and Hecke algebras of some fractal groups, Serdica Math. J. 28 (1) (2002) 47–90, math.GR/9911206.

[Bla72] N. Blackburn, Some homology groups of wreathe products, Illinois J. Math. 16 (1972) 116–129.

[Bro94] K.S. Brown, Cohomology of Groups, Springer-Verlag, New York, 1994, corrected reprint of the 1982 original.

[BŠ01] L. Bartholdi, Z. Šunić, On the word and period growth of some groups of tree automorphisms, Comm. Algebra 29 (11) (2001) 4923–4964.

[BSV99] A.M. Brunner, S.N. Sidki, A.C. Vieira, A just nonsolvable torsion-free group defined on the binary tree, J. Algebra 211 (1) (1999) 99–114.

[Bur97] W.S. Burnside, Note on the symmetric group, Proc. London Math. Soc. 28 (1897) 119–129.

[CM82] B. Chandler, W. Magnus, The History of Combinatorial Group Theory, in: Studies in the History of Mathematics and Physical Sciences, Vol. 9, Springer-Verlag, New York, 1982.

[Day57] M.M. Day, Amenable semigroups, Illinois J. Math. 1 (1957) 509–544.

[Gri84] R.I. Grigorchuk, Degrees of growth of finitely generated groups and the theory of invariant means, Izv. Akad. Nauk SSSR Ser. Mat. 48 (5) (1984) 939–985; English translation: Math. USSR-Izv. 25 (2) (1985) 259–300.

[Gri98] R.I. Grigorchuk, An example of a finitely presented amenable group that does not belong to the class EG, Mat. Sb. 189 (1) (1998) 79–100.

[Gri99] R.I. Grigorchuk, On the system of defining relations and the Schur multiplier of periodic groups generated by finite automata, in: N. Ruskuc, C.M. Campbell, E.F. Robertson, G.C. Smith (Eds.), Groups St. Andrews 1997 in Bath, I, Cambridge Univ. Press, Cambridge, 1999, pp. 290–317.

[Gri00] R.I. Grigorchuk, Just infinite branch groups, in: M.P.F. du Sautoy Dan Segal, A. Shalev (Eds.), New Horizons in pro-p Groups, Birkhäuser Boston, Cambridge, MA, 2000, pp. 121–179.

[Gup84] N.D. Gupta, Recursively presented two generated infinite p-groups, Math. Z. 188 (1) (1984) 89–90.

[Hal54] P. Hall, Finiteness conditions for soluble groups, Proc. London Math. Soc. (3) 4 (1954) 419–436.

[Hig61] G. Higman, Subgroups of finitely presented groups, Proc. Roy. Soc. London Ser. A 262 (1961) 455–475.

[Kar87] G. Karpilovsky, The Schur Multiplier, Oxford Univ. Press, New York, 1987.

[KW01] I. Kapovich, D.T. Wise, On the failure of the co-Hopf property for subgroups of word-hyperbolic groups, Israel J. Math. 122 (2001) 125–147.

[Lin73] A. Lindenmayer, Cellular automata, formal languages and developmental systems, in: Logic, Methodology and Philosophy of Science, IV, Proc. Fourth Internat. Congr., Bucharest, 1971, in: Stud. Logic Found. Math., Vol. 74, North-Holland, Amsterdam, 1973, pp. 677–691.

[LS70] R.C. Lyndon, P.E. Schupp, Combinatorial Group Theory, Springer-Verlag, 1970.

[Lys85] I.G. Lysionok, A system of defining relations for the Grigorchuk group, Mat. Zametki 38 (1985) 503–511.

[Moo97] E.H. Moore, Concerning the abstract groups of order $k!$ and $\frac{1}{2}k!$ holohedrically isomorphic with the symmetric and the alternating substitution groups on k letters, Proc. London Math. Soc. 28 (1897) 357–366.

[Neu67] H. Neumann, Varieties of Groups, Springer-Verlag, New York, 1967.

[Ol'70] A.Yu. Ol'shanskiĭ, The finite basis problem for identities in groups, Izv. Akad. Nauk SSSR Ser. Mat. 34 (1970) 376–384.

[OS01] A.Yu. Ol'shanskiĭ, M.B. Sapir, Length and area functions on groups and quasi-isometric Higman embeddings, Internat. J. Algebra Comput. 11 (2) (2001) 137–170.

[RS80] G. Rozenberg, A. Salomaa, The mathematical theory of L systems, Academic Press, New York, 1980.

[S$^+$93] M. Schönert, et al., GAP: Groups, Algorithms and Programming, RWTH, Aachen, 1993.

[Ser93] V. Sergiescu, Graphes planaires et présentations des groupes de tresses, Math. Z. 214 (3) (1993) 477–490.

[Sid87] S.N. Sidki, On a 2-generated infinite 3-group: the presentation problem, J. Algebra 110 (1) (1987) 13–23.